The Unreasonable Effectiveness of $\sqrt{-1}$

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“The Unreasonable Effectiveness of Mathematics in the Natural Sciences” [Eugene Wigner, 1960]

“It is difficult to avoid the impression that a miracle confronts us here, quite comparable in its striking nature to the miracle that the human mind can string a thousand arguments together without getting itself into contradictions, or to the two miracles of laws of nature and of the human mind’s capacity to divine them.”

Beauty in mathematics

What is the nature of beauty in maths?

$$\boxed{e^{i\pi} = -1?}$$

If that is beautiful, what is ugly?

$$\boxed{\pi^{-\frac{1}{2}s}\Gamma(\tfrac{1}{2}s)\zeta(s) = \pi^{-\frac{1}{2}(1-s)}\Gamma(\tfrac{1}{2}(1-s))\zeta(1-s)?}$$

$$\Gamma(u) = \int_0^\infty x^{u-1}e^{-x}\,dx, \qquad \zeta(s) = \sum_n \frac{1}{n^s}$$

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Beauty in the action [not the typesetting]

$$x^2 = 1 \text{ has two roots } x = \pm 1,$$
$$x^2 = -1 \text{ has none!}$$

So let's imagine one!

$$\boxed{\text{Definition: } i = \sqrt{-1}}$$

The reals $\mathbb{R}$ containing all 'real' numbers x

The complex number $\mathbb{C}$: all numbers of the form $x + iy$

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beauty and truth in mathematics

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Mathematical beauty ... arises from a combination of strangeness and inevitability. Simply defined abstractions disclose hidden quirks and complexities. Seemingly unrelated structures turn out to have mysterious correspondences. Uncanny patterns emerge, and they remain uncanny even after being underwritten by the rigor of logic." [Jim Holt]

agreed sense of right and wrong!

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Complex analysis

An "imaginary" area of maths that solves problems in

- integration
- differential equations
- number theory
- applied maths
- physics
- engineering, etc

How can this be?

Solution of the quadratic/cubic equation

Quadratic: $ax^2 + bx + c = 0,$

Solution: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Cubic: $ax^3 + bx^2 + cx + d = 0,$

Solution: ??

Conundrum: Can have three real roots given by square roots of negative numbers (without using sine, cosine etc)

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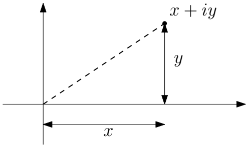
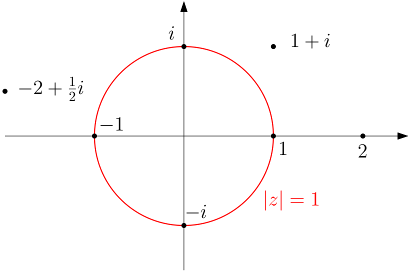
Gerolamo Cardano (1501–1576)

+ Tartaglia

The complex plane ("Argand diagram")





analysis $\longleftrightarrow$ geometry

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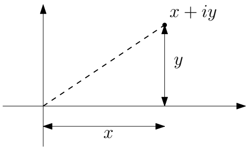

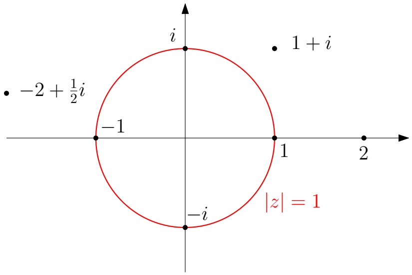

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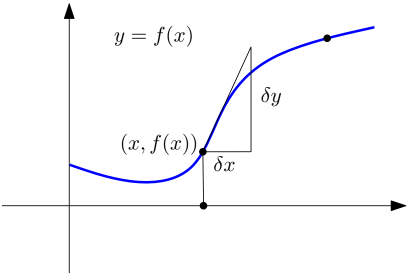
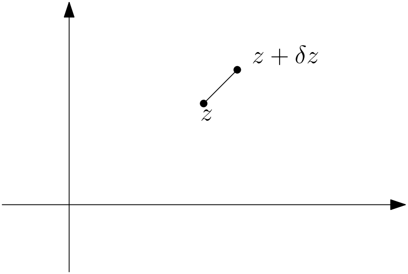
Differentiation in reals (Newton, Leibniz)



$$\text{gradient/steepness: } f'(x) = \frac{\delta y}{\delta x}$$

$$f(x + \delta x) \approx f(x) + \delta x \, f'(x)$$

Complex differentiation



there exists a complex number $\gamma = \gamma(z)$ such that:

$$f(z + \delta z) \approx f(z) + \gamma\, \delta z$$

Definition: γ is called the complex derivative of f, written $f'(z)$

Smoothness

The property "being differentiable" or "having a derivative" is a smoothness property

Complex differentiability is a much more severe restriction than real differentiability

Therefore, "fewer" complex functions are differentiable

Therefore, they have more properties

They are called analytic or holomorphic

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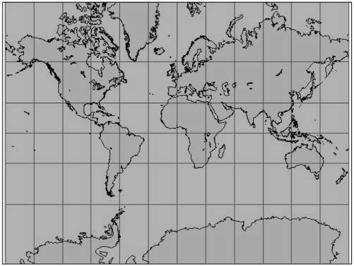
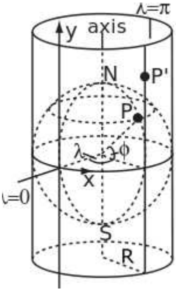
Conformality





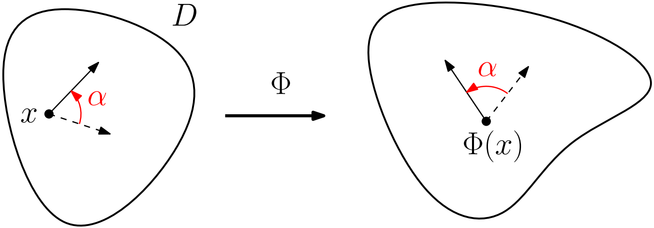
Mercator projection is conformal

It maps the sphere to the cylinder and preserves angles

"Theorem" analytic = conformal

Conformal functions

$\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ is conformal on the domain D if it preserves angles



1. Conformal maps are locally a dilation + rotation
2. Conformal = analytic on $D \subseteq \mathbb{C}$ with non-zero derivative

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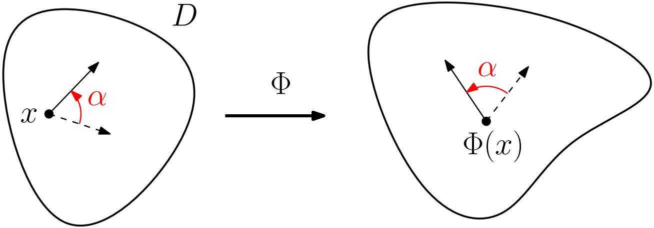

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Universality

Physical observable	ice melts
Model of statistical physics	lattice model
Parameter	temperature
Phase transition	melting
What sort of singularity is it?	universality

Recent progress with universality

Physics says: universality is connected to renormalization
Looking on bigger and bigger scales

Progress depends heavily on the number of dimensions

Exceptional case of two dimensions

universality $\leftrightarrow$ invariance under local rotations/dilations
$\leftrightarrow$ conformal invariance
$\leftrightarrow$ complex analysis
$\leftrightarrow$ conformal field theory

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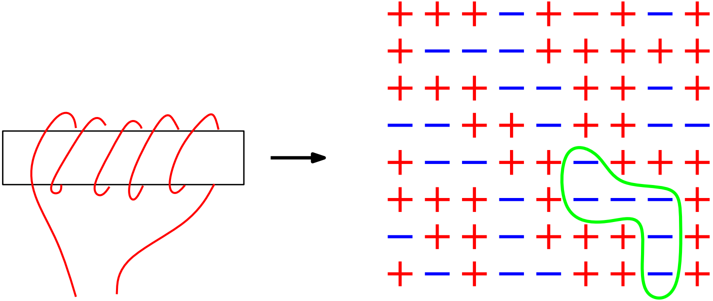
Ising model for ferromagnet



Hamiltonian: configuration σ has (large) energy

$$H(\sigma) = \text{total length of interfaces}$$

probability of configuration $Ce^{-H(\sigma)/T}$

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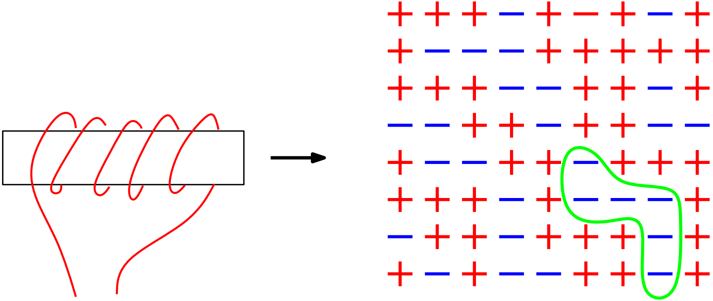

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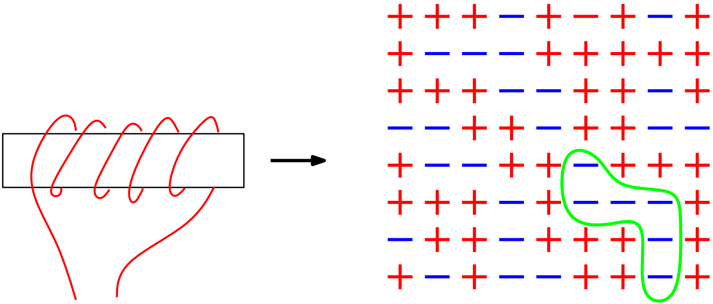

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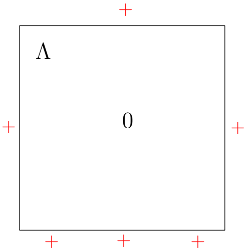
Phase transition in infinite volume

finite $\Lambda \subseteq \mathbb{Z}^2$



'order parameter': $M_\Lambda(\beta) = P_\beta^+(\sigma_0 = +1)$

'thermodynamic limit': $M_\Lambda(\beta) \downarrow M(\beta)$ as $\Lambda \uparrow \mathbb{Z}^2$

Theorem: $M(\beta) \begin{cases} = 0 & \text{if } \beta < \beta_c, \\ > 0 & \text{if } \beta > \beta_c, \end{cases}$

where $2\beta_c = \log(1 + \sqrt{2})$.

Lenz/Ising, Onsager, …, Chelkak/Smirnov

critical exponents

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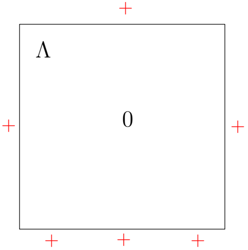

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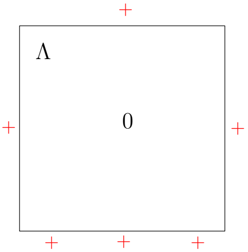

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Question: is the 'type of singularity' the same for all two-dimensional systems?

Answer (Chelkak–Smirnov): Yes, for very large class of systems.

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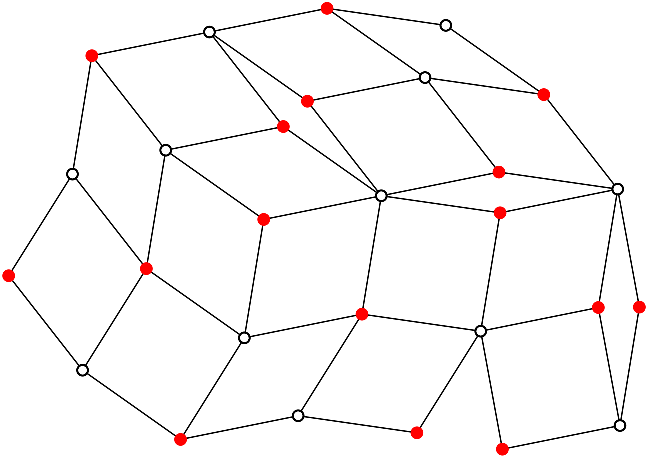
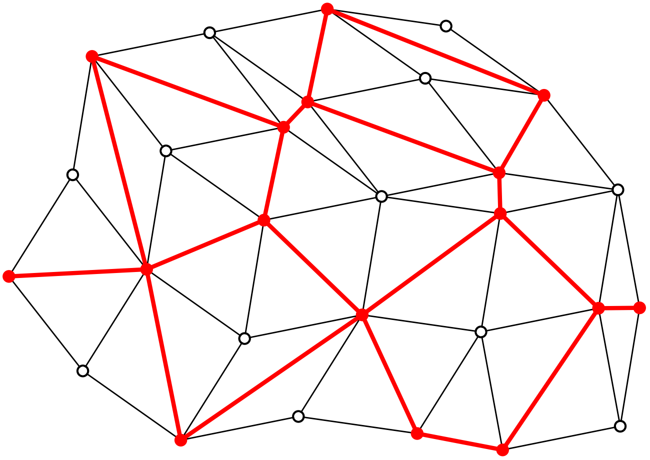
Rhombic tiling



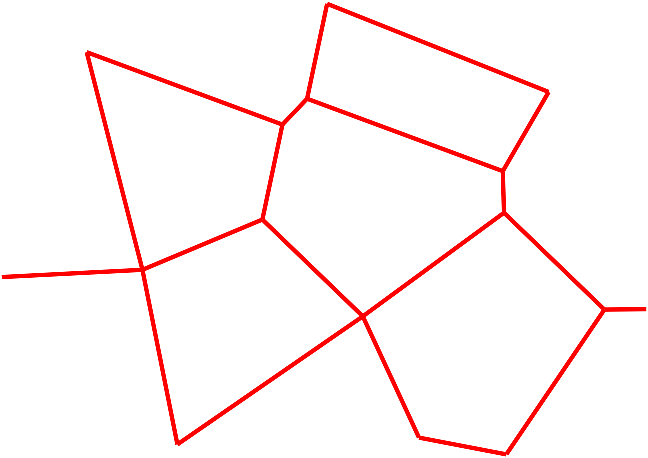
Penrose, de Bruijn

Rhombic tiling + isoradial graph



Duffin +

An isoradial graph



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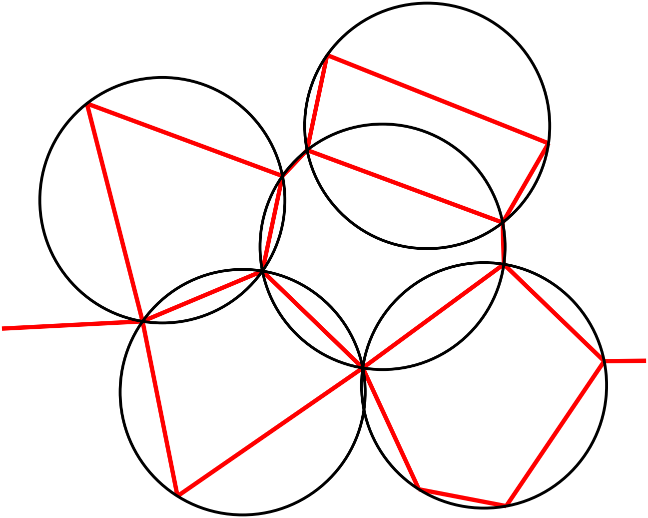
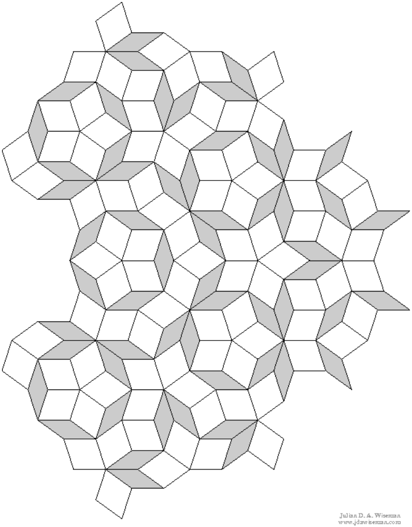
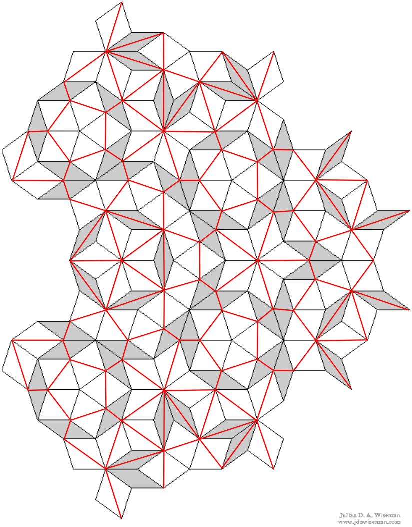
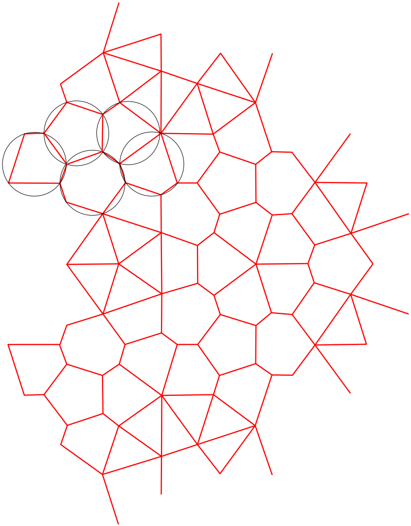


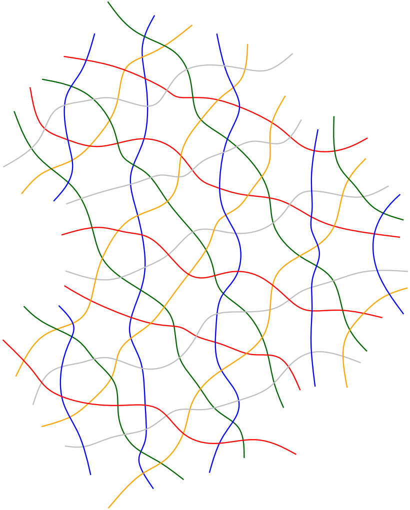
Penrose percolation





Penrose: isoradial graph and track-system





What's going on?

1. isoradial graphs are exactly those for which one can construct a discrete complex analysis [Duffin 1968].
2. discrete $\rightarrow$ continuum
3. limit object is invariant under conformal maps.

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What's next?